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The height measure of p-adic balls

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Abstract

In this paper we give the height measure of p-adic balls. In other words, given any $x \in \mathbf{Q}_p$, we give the chance that a random rational number r satisfies $|r - x|_p \leqslant \varepsilon$.

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Let $H(m/n) = \max\{|m|, n\}$ be the *height* of the rational number m/n, where $m \in \mathbf{Z}$, $n \in \mathbf{N}$ and $\gcd(m, n) = 1$. Given $U \subseteq \mathbf{Q}$, consider the limit

$$\lim_{t \to \infty} \frac{\#\{r \in U : H(r) \leqslant t\}}{\#\{r \in \mathbf{Q} : H(r) \leqslant t\}}.$$

If it exists, we denote its value $\mu(U)$: the *height density* of U.

More in general, if U is a subset of a completion K of $\mathbf{Q}$ and the limit

$$\mu(U) = \lim_{t \to \infty} \frac{\#\{r \in \bar{U} \cap \mathbf{Q} : H(r) \leq t\}}{\#\{r \in \mathbf{Q} : H(r) \leq t\}} \tag{1}$$

exists, we say that U is μ-measurable and call $\mu(U)$ the *height measure* of U.

In [3] we proved that any interval $(a, b) \subset \mathbf{R}$ is μ-measurable and gave a simple formula for its value. If p is a finite place of $\mathbf{Q}$, denote as usual v_p the associated valuation, $|\cdot|_p$ the norm and $\mathbf{Q}_p$ the completion. Let $B(x, p^{-e}) = \{r \in \mathbf{Q}_p : |x - r|_p \leqslant p^{-e}\}$ be the (closed) p-adic *ball* of centre x and radius p^{-e}. In this paper we prove that:

Theorem 1. *For any $x \in \mathbf{Q}_p$, the p-adic ball $B(x, p^{-e})$ is μ-measurable. Moreover:*

- *if $e \leqslant v(x)$:*

$$\mu\big(B(x, p^{-e})\big) = \begin{cases} \dfrac{p^{1-e}}{p+1} & \text{if } e \geqslant 0, \\[2ex] 1 - \dfrac{p^e}{p+1} & \text{if } e \leqslant 0; \end{cases}$$

- *while, if $e > v(x)$:*

$$\mu\big(B(x, p^{-e})\big) = \begin{cases} \dfrac{p^{1-e}}{p+1} & \text{if } v(x) \geqslant 0 \\[2ex] \dfrac{p^{1-e}}{|x|_p^2\,(p+1)} & \text{if } v(x) < 0 \end{cases}$$

In particular,

$$\mu(\{r \in \mathbf{Q} : v(r-x) = e\}) = \begin{cases} p^{-|e|}\dfrac{p-1}{p+1} & \text{if } e \leqslant v(x) \text{ or } e \geqslant 0; \\[2ex] \dfrac{p}{p+1}\left(\dfrac{p}{|x|_p^2} - 1\right) & \text{if } v(x) < e = -1; \\[2ex] \dfrac{p^{-e}}{|x|_p^2}\dfrac{p-1}{p+1} & \text{if } v(x) < e < -1. \end{cases}$$

1 A dutiful note on measure theory

Unfortunately, Eq. (1) does not define what is usually called a measure: indeed, $\mathbf{Q}$ is not σ-additive, so no function on it may be σ-additive. For example, $\mathbf{Q} = \bigcup_{r \in \mathbf{Q}}\{r\}$, but

$$1 = \mu(\mathbf{Q}) \neq \sum_{r \in \mathbf{Q}} \mu(\{r\}) = 0.$$

This is not a serious problem, since we can easily define a real (pun intended!) measure on $\mathbf{Q}_p$ which agrees, on p-adic balls, with our definition:

Definition. Let p be a finite or infinite place of $\mathbf{Q}$. For any $E \subset \mathbf{Q}_p$ and $\delta > 0$, let

$$\mu_\delta(E) = \inf_{\substack{|B_i| \leqslant \delta \\ \bigcup B_i \supset E}} \sum \mu(B_i),$$

where the B_i are p-balls and the unions are countable. Furthermore, let

$$\mu^*(E) = \sup_{\delta > 0} \mu_\delta(E).$$

Theorem 2. *For any place p of $\mathbf{Q}$, the set function μ^* is a σ-additive measure on $\mathbf{Q}_p$, the Borel sets are measurable and $\mu^*(B) = \mu(B)$ for any p-ball.*

Proof. Recall that μ is an additive set function by theorem 4 of [3], hence by general measure theory on metric spaces (see for example theorem 23 of [4]) μ^* is a σ-additive measure on $\mathbf{Q}_p$ and the Borel sets are measurable.

We are left to prove that μ^* coincides with μ on p-balls. If $p = \infty$, since by theorem 4 of [3] $\mu(B)$ is essentially the length of the interval B, this is a classical result: see, for example, §5 of [4]. Suppose now that $p < \infty$: we claim that

$$\mu_\delta(B) = \mu(B), \qquad \text{for every } \delta > 0. \tag{2}$$

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Fix such a δ, and let $\{B_i\}_{i \in I}$ be p-balls with $|B_i| \leqslant \delta$ and $\bigcup B_i \supset B$. Since B is compact and each B_i is open, we may suppose that I is finite. For every $i \in I$, we may clearly suppose that $B_i \cap B \neq \emptyset$; if $x_i \in B_i \cap B$, then we can take x_i to be the centre of both; so either $B_i \subset B$ or $B_i \supset B$, the latter being an uninteresting case. Similarly, we may assume that the B_i are all pairwise disjoint. Therefore, we have $B = \bigcup B_i$ where the union is finite and disjoint; equation (2) now follows from the additivity of μ. □

2 Preliminary results

From now on we fix a finite prime p.

Definition. In analogy to Euler's φ function, we define for any positive integer t and any positive number x, a function

$$\varphi(t, x) = \#\{\text{positive integers} \leqslant x \text{ which are relatively prime to } t\}$$

We proved in [3] that:

Proposition 3. *Denote $d(n)$ the number of divisors of n. Then, for any $x, t > 0$ we have that*

$$\varphi(t, x) = \frac{x}{t}\varphi(t) \pm d(t),$$

where $a = b \pm \delta$ means that $|a - b| \leqslant \delta$.

Lemma 4. *Suppose t, a, e are integer numbers with $t > 1$ and fix $T > 0$. Then*

$$\#\{n \in \mathbf{Z} : 0 \leqslant n \leqslant T, \gcd(n, t) = 1, \, v(n - at) \geqslant e\}$$
$$= \begin{cases} p^{-\max\{0,e\}}\frac{T}{t}\varphi(t) \pm 2d(t) & \text{if } p \nmid t, \\ \frac{T}{t}\varphi(t) \pm d(t) & \text{if } p \mid t \text{ and } e \leqslant 0, \\ 0 & \text{if } p \mid t \text{ and } e > 0. \end{cases}$$

Proof. Clearly, if $e < 0$, we may replace the condition $v(n - at) \geqslant e$ with $v(n - at) \geqslant 0$; in other words, we may suppose e non negative.

Suppose $p \mid t$: if $e > 0$, the statement is obvious; if $e = 0$, it follows from Proposition 3. Suppose now that $p \nmid t$. Then

$$\#\{n \in \mathbf{Z} : 0 \leqslant n \leqslant T, \, \gcd(n, t) = 1, \, v(n - at) \geqslant e\}$$
$$= \#\left\{ \begin{matrix} n = p^e n' + at : n' \in \mathbf{Z}, \, -\dfrac{at}{p^e} \leqslant n' \leqslant \dfrac{T - at}{p^e}, \\ \gcd(p^e n' + at, t) = 1, \, v(p^e n') \geqslant e \end{matrix} \right\}$$
$$= \#\left\{ n' \in \mathbf{Z} : -\dfrac{at}{p^e} \leqslant n' \leqslant \dfrac{T - at}{p^e}, \, \gcd(n', t) = 1 \right\}. \tag{3}$$

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Suppose $a \geqslant 0$ and $T \geqslant at$, then equation (3) becomes

$$= \varphi \left(t, \frac{T - at}{p^e} \right) + \varphi \left(t, \frac{at}{p^e} \right);$$

by proposition 3, this is

$$= p^{-e} \frac{T}{t} \varphi(t) \pm 2d(t). \tag{4}$$

If $T < at$ or $a < 0$, it is a trivial calculation to verify that equation (4) still holds true. $\quad\square$

Remark 5. The Lemma holds even if $a \in \mathbf{Z}_p$: it suffices to write $a = a' + O(p^\eta)$ with $a' \in \mathbf{Z}$ and η large enough so that, for every positive $n \leqslant T$, $v(n - at) = v(n - a't)$. In particular, it holds if $a \in \mathbf{Q}$ with $v_p(a) \geqslant 0$.

Lemma 6. *Suppose m and n are relatively prime integers. Then*

$$v(m/n) \geqslant e \qquad \text{if and only if} \qquad \begin{cases} v(m) \geqslant e & \text{if } e > 0, \\ v(n) \leqslant -e & \text{if } e \leqslant 0. \end{cases}$$

Proof. Obvious. $\quad\square$

Proposition 7. *For any $T > 0$ we have*

$$\sum_{n \leqslant T} \varphi(n) = \frac{1}{2\zeta(2)} T^2 + O(T \log T), \qquad \sum_{n \leqslant T} d(n) = T \log T + O(T),$$

$$\sum_{\substack{n \leqslant T \\ p|n}} \varphi(n) = \frac{1}{2\zeta(2)(p+1)} T^2 + o(T^2), \qquad \sum_{\substack{n \leqslant T \\ p \nmid n}} \varphi(n) = \frac{p}{2\zeta(2)(p+1)} T^2 + o(T^2).$$

Proof. The first and second assertions are very well known (see, e.g., [1, Chapter 3]) while the third clearly follows from the last one. Let

$$\varphi'(n) = \begin{cases} \varphi(n) & \text{if } p \nmid n, \\ 0 & \text{if } p \mid n. \end{cases}$$

Since φ', as well as φ, is multiplicative, we have for $\mathrm{Re}(s) > 2$:

$$\sum_{\substack{n=1 \\ p \nmid n}}^{\infty} \frac{\varphi(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\varphi'(n)}{n^s} = \frac{p^s - p}{p^s - 1} \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s} = \frac{p^s - p}{p^s - 1} \cdot \frac{\zeta(s-1)}{\zeta(s)}.$$

In particular, the LHS is regular on the line $\mathrm{Re}(s) = 2$ with the exception of a pole of first order at $s = 2$ with residue $p/(p+1)\zeta(2)$. Thanks to the Tauberian theorem for Dirichlet series (see [2, XV, §3]) it follows that, after a change of coordinate $s \mapsto s - 1$,

$$\sum_{n=1}^{T} \frac{\varphi'(n)}{n} = \frac{p}{p+1} \cdot \frac{T}{\zeta(2)} + o(T).$$

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Applying the following "integration" lemma, we get the proposition. Note that, actually, the error term for the two last assertions is $O\left(T\log T\right)$: since we will not need this improved estimate, we will content ourselves with the simpler previous proof. $\qquad\square$

Lemma 8. *Let* $\{b_n\}_{n\in\mathbf{N}}$ *be complex numbers and let* $B(T) = \sum_{n=1}^{T} b_n$. *Suppose that there is* $\beta \in \mathbf{C}$ *such that* $B(T) = \beta T + o\left(T\right)$. *Then*

$$\sum_{n=1}^{T} nb_n = \frac{\beta}{2}T^2 + o\left(T^2\right).$$

3 Slices

In order to prove Theorem 1, we count how many rational points in $B(x, p^e)$ have a given height, then we use the previous Lemma to sum over all heights.

Definition. For any $x \in \mathbf{Q}_p$ and $e \in \mathbf{Z}$, let $B(x, p^e; T) = B(x, p^e) \cap \mathbf{Q}(T)$, where $\mathbf{Q}(T) = \{t \in \mathbf{Q} : H(t) = T\}$.

Lemma 9. *For every positive integer* T: $\#\mathbf{Q}(T) = 4\varphi(T)$.

Proof. Obvious $\qquad\square$

Proposition 10. *Fix* $x \in \mathbf{Q}_p$ *and* $e \in \mathbf{Z}$ *such that* $v(x) \geqslant e$, *then for any* $t \in \mathbf{Z}^{>0}$ *we have:*

1. *if* $e > 0$,
$$\#B(x, p^{-e}; t) = \begin{cases} 2p^{-e}\varphi(t) \pm 4d(t) & \text{if } v(t) = 0, \\ 0 & \text{if } 0 < v(t) < e, \\ 2\varphi(t) & \text{if } v(t) \geqslant e; \end{cases}$$

2. *if* $e \leqslant 0$,
$$\#B(x, p^{-e}; t) = \begin{cases} 2\left(2 - p^{e-1}\right)\varphi(t) \pm 4d(t) & \text{if } v(t) = 0, \\ 4\varphi(t) & \text{if } 0 < v(t) < 1 - e, \\ 2\varphi(t) & \text{if } v(t) \geqslant 1 - e. \end{cases}$$

Proof. Since $0 \in B(x, p^{-e})$, we have $B(x, p^{-e}) = B(0, p^{-e})$ and thus, for any t, $B(x, p^{-e}; t) = B(0, p^{-e}; t)$. Therefore, we may as well suppose that $x = 0$. Write $B(0, p^{-e}; t)$ as

$$\left\{ \frac{m}{n} : (m, n) \in \mathbf{Z} \times \mathbf{Z}^{>0}, \ \max\{|m|, n\} = t, \ \gcd(m, n) = 1, \ v(m/n) \geqslant e \right\}. \qquad (5)$$

Suppose $e > 0$: then (5) becomes, using Lemma 6,

$$B(0, p^{-e}; t)$$
$$= \{m/n : (m, n) \in \mathbf{Z} \times \mathbf{Z}^{>0}, \ \max\{|m|, n\} = t, \ \gcd(m, n) = 1, \ v(m) \geqslant e\}$$
$$= \{\pm t/n : n \in \mathbf{Z}, \ 1 \leqslant n \leqslant t, \ \gcd(n, t) = 1, \ v(t) \geqslant e\}$$
$$\qquad \cup \ \{m/t : m \in \mathbf{Z}, \ -t \leqslant m \leqslant t, \ \gcd(m, t) = 1, \ v(m) \geqslant e\}.$$

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The first part of the proposition now follows from Lemma 4.

In order to prove the second part, notice that

$$\mathbf{Q}(T) = B(0, p^{-e}; t) \cup \{r \in \mathbf{Q} : H(r) = T, \ v(r) \leqslant e - 1\}$$

and that $r \mapsto 1/r$ induces a 1-to-1 correspondence

$$\{r \in \mathbf{Q} : H(r) = T, \ v(r) \leqslant e - 1\} \longleftrightarrow \{r \in \mathbf{Q} : H(r) = T, \ v(r) \geqslant 1 - e\}$$

Thus $\#B(0, p^{-e}; t) = \#\mathbf{Q}(T) - \#B(0, p^{-(1-e)}; t)$ and part (2) follows from part (1) and Lemma 9. □

Proposition 11. *Fix* $x \in \mathbf{Q}_p$ *and* $e \in \mathbf{Z}$ *such that* $v(x) < e$, *then for any* $t \in \mathbf{Z}^{>0}$ *we have:*

1. *if* $v(x) > 0$,

$$\#B(x, p^{-e}; t) = \begin{cases} 2p^{-e}\varphi(t) \pm 4d(t) & \text{if } v(t) = 0, \\ 2\dfrac{p^{1+v(x)-e}}{p-1}\varphi(t) \pm 4d(t) & \text{if } v(t) = v(x), \\ 0 & \text{otherwise;} \end{cases}$$

2. *if* $v(x) = 0$,

$$\#B(x, p^{-e}; t) = \begin{cases} 4p^{-e}\varphi(t) \pm 8d(t) & \text{if } v(t) = 0, \\ 0 & \text{if } v(t) \neq 0; \end{cases}$$

3. *if* $v(x) < 0$,

$$\#B(x, p^{-e}; t) = \begin{cases} 2p^{2v(x)-e}\varphi(t) \pm 4d(t) & \text{if } v(t) = 0, \\ 2\dfrac{p^{1+v(x)-e}}{p-1}\varphi(t) \pm 4d(t) & \text{if } v(t) = -v(x), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have that

$$B(x, p^{-e}; t)$$
$$= \left\{ \begin{array}{c} m/n : (m, n) \in \mathbf{Z} \times \mathbf{Z}^{>0}, \ \max\{|m|, n\} = t, \ \gcd(m, n) = 1, \\ v\left(\dfrac{m}{n} - x\right) \geqslant e \end{array} \right\}$$
$$= \left\{ \pm t/n : n \in \mathbf{Z}, \ 1 \leqslant n \leqslant t, \ \gcd(n, t) = 1, \ v\left(\dfrac{\pm t}{n} - x\right) \geqslant e \right\}$$
$$\bigcup \left\{ m/t : m \in \mathbf{Z}, \ -t \leqslant m \leqslant t, \ \gcd(m, t) = 1, \ v\left(\dfrac{m}{t} - x\right) \geqslant e \right\}.$$

Let us call these two sets respectively B_1 and B_2.

Consider B_2. We have

$$v\left(\frac{m}{t} - x\right) \geqslant e \iff v(m - xt) \geqslant e + v(t). \tag{6}$$

Suppose $v(t) = 0$.

- If $v(x) \geqslant 0$, we can apply Lemma 4: since $e > v(x)$, we get $\#B_2 = 2p^{-e}\varphi(t) \pm 4d(t)$.

- If $v(x) < 0$, we get $v(m - xt) = v(x) < e$, hence Eq. (6) is false and $\#B_2 = 0$.

Suppose now $v(t) > 0$. Then $\gcd(m, t) = 1$ implies that $v(m) = 0$.

- If $v(xt) > 0$, then $v(m - xt) = v(m) = 0$ with $0 < v(x) + v(t) < e + v(t)$. Hence Eq. (6) is false.

- If $v(xt) = 0$, let $\eta = v(t) > 0$, $x' = xp^\eta$ and $t' = tp^{-\eta}$. Hence

$$\#B_2 = \#\{m : -t \leqslant m \leqslant t, \gcd(m, t) = 1, v(m - x't') \geqslant e + \eta\}.$$

We can replace the condition $\gcd(m, p^\eta t') = 1$ with $\gcd(m, t') = 1$, since $p \mid m$ would imply $v(m - x't') = 0 = v(x) + v(t) < e + v(t)$. Lemma 4 yields

$$\#B_2 = 2p^{-e-\eta}\frac{t}{t'}\varphi(t') \pm 4d(t) = 2\frac{p^{1-e-\eta}}{p-1}\varphi(t) \pm 4d(t).$$

- If $v(xt) < 0$, then $v(m - xt) = v(xt) = v(x) + v(t) < e + v(t)$; Eq. (6) is therefore false.

Putting everything together, we have

$$\#B_2 = \begin{cases} 2p^{-e}\varphi(t) \pm 4d(t) & \text{if } v(t) = 0 \text{ and } v(x) \geqslant 0, \\ 2\frac{p^{1-e-v(t)}}{p-1}\varphi(t) \pm 4d(t) & \text{if } v(t) > 0 \text{ and } v(x) = -v(t), \\ 0 & \text{otherwise.} \end{cases}$$

Consider now B_1. Write $x = x'p^\eta$ with $x' \in \mathbf{Z}_p$ and $\eta = v(x)$. Since $v(x) < e$ and $v(\pm t/n - x) \geqslant e$ we have

$$v(t/n) = v(x) = \eta, \quad \text{with the constraint } \gcd(t, n) = 1 \tag{7}$$

Assume that $\eta \geqslant 0$, then Eq. (7) implies that $v(t) = \eta$ and $v(n) = 0$; in particular $B_1 = \emptyset$ if $v(t) \neq v(x)$. Suppose thus $v(t) = \eta$ and write $t = t'p^\eta$. Then

$$v\left(\pm\frac{t}{n} - x\right) = \eta + v\left(nx' \mp t'\right) = \eta + v(n \mp x'^{-1}t')$$

and, since $v(n) = 0$,

$$B_1 = \left\{\pm t/n : n \in \mathbf{Z}, \ 1 \leqslant n \leqslant t, \ \gcd(n, t') = 1, \ v(n \mp x'^{-1}t') \geqslant e - \eta\right\}.$$

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It follows, by lemma 4, that

$$\#B_1 = 2p^{\eta-e}\frac{t}{t'}\varphi(t') \pm 4d(t) = \begin{cases} 2\dfrac{p^{1+\eta-e}}{p-1}\varphi(t) \pm 4d(t) & \text{if } v(t) = v(x) > 0, \\ 2p^{-e}\varphi(t) \pm 4d(t) & \text{if } v(t) = v(x) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Assume now that $\eta < 0$, then equation (7) implies that $v(t) = 0$ and $v(n) = -\eta$; in particular, $B_1 = \emptyset$ if $v(t) > 0$. Suppose not and write $n = n'p^{-\eta}$. Then

$$v\left(\pm\frac{t}{n} - x\right) = \eta + v(\pm t - n'x').$$

Since $e - \eta > 0$ and $v(x'^{-1}t) = 0$, we have

$$\#B_1 = \#\left\{ \begin{array}{c} \pm t/n'p^{-\eta} : n' \in \mathbf{Z}, \ 1 \leqslant n' \leqslant p^\eta t, \ \gcd(n', t) = 1, \\ v(n' \mp x'^{-1}t) \geqslant e - \eta \end{array} \right\}$$

$$= 2p^{\eta-e}\frac{p^\eta t}{t}\varphi(t) \pm 4d(t) = 2p^{2\eta-e}\varphi(t) \pm 4d(t).$$

The Proposition follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

4 Proof of Theorem 1

Let e be a strictly positive integer. Then, by Proposition 7,

$$\sum_{\substack{t \leqslant T \\ p^e|t}} \varphi(t) = p^{e-1}\sum_{\substack{t \leqslant T/p^{e-1} \\ p|t}} \varphi(t) = \frac{p^{1-e}}{2\zeta(2)(p+1)}T^2 + O\left(T\log T\right). \qquad (8)$$

It follows that, the sum being over positive terms,

$$\sum_{\substack{t \leqslant T \\ v(t)=e}} \varphi(t) = \sum_{\substack{t \leqslant T \\ p^e|t}} \varphi(t) - \sum_{\substack{t \leqslant T \\ p^{e+1}|t}} \varphi(t) = p^{-e}\frac{p-1}{2\zeta(2)(p+1)}T^2 + O\left(T\log T\right). \qquad (9)$$

Suppose now that $0 < e \leqslant v(x)$. Then Proposition 10 yields

$$\sum_{t \leqslant T} \#B(x, p^{-e}; t) = \sum_{\substack{t \leqslant T \\ p\nmid t}} \left(\frac{2}{p^e}\varphi(t) \pm 4d(t)\right) + \sum_{\substack{t \leqslant T \\ p^e|t}} 2\varphi(t)$$

$$= \frac{2p^{1-e}}{2\zeta(2)(p+1)}T^2 + \frac{2p^{1-e}}{2\zeta(2)(p+1)}T^2 + O\left(T\log T\right)$$

$$= \frac{4p^{1-e}}{2\zeta(2)(p+1)}T^2 + O\left(T\log T\right);$$

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therefore
$$\mu\big(B(x,p^{-e})\big) = \lim_{T \to +\infty} \frac{\sum_{t \leqslant T} \#B(x,p^{-e};t)}{\sum_{t \leqslant T} \#\mathbf{Q}(t)} = \frac{p^{1-e}}{p+1}.$$

If $e \leqslant 0$ and $e \leqslant v(x)$ we have, instead:

$$\sum_{t \leqslant T} \#B(x,p^{-e};t) = \sum_{t \leqslant T} 4\varphi(t) - 2 \sum_{\substack{t \leqslant T \\ p \nmid t}} \big(p^{e-1}\varphi(t) \pm 4d(t)\big) - 2 \sum_{\substack{t \leqslant T \\ p^{1-e} | t}} \varphi(t)$$

$$= \frac{4}{2\zeta(2)} T^2 - \frac{2p^e}{2\zeta(2)(p+1)} T^2 - \frac{2p^e}{2\zeta(2)(p+1)} T^2 + O\left(T \log T\right)$$

$$= 4 \left(1 - \frac{p^e}{p+1}\right) \frac{1}{2\zeta(2)} T^2 + O\left(T \log T\right).$$

Thus
$$\mu\big(B(x,p^{-e})\big) = 1 - \frac{p^e}{p+1}.$$

Suppose now $e > v(x) > 0$. Then Proposition 11 and Eq. (9) yield

$$\sum_{t \leqslant T} \#B(x,p^{-e};t) = \sum_{\substack{t \leqslant T \\ p \nmid t}} \left(\frac{2}{p^e}\varphi(t) \pm 4d(t)\right) + \sum_{\substack{t \leqslant T \\ v(t)=v(x)}} \left(2\frac{p^{v(x)-e}}{1-1/p}\varphi(t) \pm 4d(t)\right)$$

$$= \frac{4p^{1-e}}{2\zeta(2)(p+1)} T^2 + O\left(T \log T\right);$$

so that $\mu\big(B(x,p^{-e})\big) = p^{1-e}/(p+1)$. If $e > v(x) = 0$, the calculation clearly gives the same result. Suppose at last that $e > v(x)$ with $v(x) < 0$. Then

$$\sum_{t \leqslant T} \#B(x,p^{-e};t) = \sum_{\substack{t \leqslant T \\ p \nmid t}} \left(2p^{2v(x)-e}\varphi(t) \pm 4d(t)\right)$$

$$+ \sum_{\substack{t \leqslant T \\ v(t)=-v(x)}} \left(2\frac{p^{1+v(x)-e}}{p-1}\varphi(t) \pm 4d(t)\right)$$

$$= \frac{4p^{1+2v(x)-e}}{2\zeta(2)(p+1)} T^2 + O\left(T \log T\right);$$

hence $\mu\big(B(x,p^{-e})\big) = p^{1+2v(x)-e}/(p+1)$. $\qquad\square$

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